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Generalized Burnett Hydrodynamics

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Abstract Equations of hydrodynamics (derived from the Boltzmann equation) beyond the Navier-Stokes level are studied by a method proposed earlier by the author. The main question we consider is the following: What is the most natural replacement for classical (ill-posed) Burnett equations?

It is shown that, in some sense, it is a two-parameter set of Generalized Burnett Equations (GBEs) derived in this paper. Some equations of this class are even simpler than original Burnett equations. The region of stability in the space of parameters and other properties of GBEs are discussed.

Keywords Boltzmann equation · Chapman-Enskog method · Burnett equations · Hyperbolicity · Perturbation theory · Hydrodynamics

1 Introduction

We continue in this paper to study equations of hydrodynamics (derived from the Boltzmann equation) beyond the Navier-Stokes level. The classical sequence (Euler equations, Navier-Stokes equations, Burnett equations, etc.) actually breaks down at the Burnett level since Burnett equations are ill-posed [1]. This phenomenon is discussed in detail in our previous paper [2]. Different approaches to deal with this problem were proposed by several authors (see [7, 8] for review). In particular, we mention the papers by Slemrod [6, 7] and by Jin and Slemrod [4]. An alternative general method of regularization of various asymptotic expansions was proposed in [2]. In order to make this paper self-consistent we remind briefly the main idea of the method.

We consider a general evolution equation for an abstract vector $x(t)$

$$x_t = T(x;\varepsilon) = A(x) + \varepsilon B(x) + \varepsilon^2 C(x) + \cdots, \tag{1}$$

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where A, B, C, ... are time-independent differentiable nonlinear operators, $\varepsilon > 0$ is a small parameter In our case $x(t)$ is the vector of hydrodynamical variables, whereas $A(x)$, $B(x)$ and $C(x)$ are Euler, Navier-Stokes and Burnett operators respectively; ε is the usual Knudsen number. The difficulty is that we cannot simply neglect terms of higher orders $n \geq 3$ in ε, since the resulting "Burnett equations" are ill-posed.

This is, however, true provided we stay in a fixed coordinate system in the space of the x-variable. On the other hand, we can admit "small" (formally invertible for $\varepsilon \to 0$) changes of coordinates, for example,

$$y = x + \varepsilon^2 R(x) \quad \Rightarrow \quad x = y - \varepsilon^2 R(y) + \cdots, \tag{2}$$

with an arbitrary time-independent differentiable operator R. Then we obtain the following equation for $y(t)$:

$$\begin{aligned} y_t &= \widetilde{T}(y;\varepsilon) = A(y) + \varepsilon B(y) + \varepsilon^2 \widetilde{C}(y) + \cdots, \\ \widetilde{C}(y) &= C(y) + [R, A](y), \quad [R, A](y) = R'_y A(y) - A'_y R(y), \end{aligned} \tag{3}$$

where R'_y and A'_y are the Fréchet derivatives (linear operators) of $R(y)$ and $A(y)$ respectively. Note that $[R, A]$ is the usual commutator in case of linear operators R and A. Hence, the truncated at the "Burnett level" $O(\varepsilon^2)$ equations are, generally speaking, not unique. They depend on an arbitrary operator R. One can hope that there exist such operators R that make the resulting equations well-posed. Indeed the first example of such operator R for classical Burnett equations was constructed in [2]. The resulting Hyperbolic Burnett Equations (HBEs) are well-posed and satisfy the linearized H-theorem.

On the other hand, HBEs are just one example of "stable" Burnett equations regularized by this method. They are far not perfect: too complicated, do not have the standard form of hydrodynamic equations.

Can we find something better? What is the most natural replacement for classical (ill-posed) Burnett equations? How to treat the obvious non-uniqueness of equations of hydrodynamics (even at the Navier-Stokes level) related to possible changes of coordinates? We try to address these and similar questions below.

2 Equations of Hydrodynamics and Their Transformation

We assume that the distribution function $f(x, v, t)$, where $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$ and $t > 0$ correspond respectively to position, velocity and time, satisfy the scaled Boltzmann equation [3]

$$f_t + v \cdot f_x = \frac{1}{\varepsilon} Q(f, f), \tag{4}$$

where $\varepsilon > 0$ denotes the Knudsen number. Hydrodynamic variables (the density ρ, the bulk velocity $u \in \mathbb{R}^3$ and the temperature T) are defined by equalities

$$\rho = \langle f \rangle, \quad \rho u = \langle f, v \rangle, \quad \rho T = \frac{1}{3} \langle f, |c|^2 \rangle, \quad c = v - u, \tag{5}$$

in the notation

$$\langle f, g \rangle = \langle fg \rangle = \int_{\mathbb{R}^3} dv f(v) g(v). \tag{6}$$

Usual conservation laws lead to general (non-closed) equations of hydrodynamics [3]

$$\begin{aligned} &\rho_t + \operatorname{div} \rho u = 0, \qquad \rho \mathcal{D} u_\alpha + \frac{\partial p}{\partial x_\alpha} + \frac{\partial \pi_{\alpha\beta}}{\partial x_\beta} = 0, \\ &\frac{3}{2}\rho \mathcal{D} T + p \operatorname{div} u + \pi_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} + \operatorname{div} q = 0, \end{aligned} \tag{7}$$

where

$$p = \rho T, \qquad \mathcal{D} = \partial_t + u \cdot \partial_x, \qquad \pi_{\alpha\beta} = \left\langle f, c_\alpha c_\beta - \frac{1}{3}\delta_{\alpha\beta}|c|^2 \right\rangle, \qquad q_\alpha = \frac{1}{2}\left\langle f, c_\alpha |c|^2 \right\rangle, \tag{8}$$
$$\alpha, \beta = 1, 2, 3.$$

The usual rule of summation over repeated indexes is assumed here and below. Assuming that

$$f = \rho M + \varepsilon F, \quad M = (2\pi T)^{-3/2} e^{-\frac{|c|^2}{2T}}, \ F = F_0 + \varepsilon F_1 + \cdots, \tag{9}$$

where $\rho(x,t;\varepsilon)$, $u(x,t;\varepsilon)$ and $T(x,t;\varepsilon)$ are "true" hydrodynamic moments of f, we obtain the formal expansion

$$\begin{aligned} \pi_{\alpha\beta} &= \varepsilon\langle F, c_\alpha c_\beta\rangle = \varepsilon \pi_{\alpha\beta}^{N-S} + \varepsilon^2 \pi_{\alpha\beta}^{B} + \cdots, \\ q_\alpha &= \frac{\varepsilon}{2}\langle F, |c|^2 c_\alpha\rangle = \varepsilon q_\alpha^{N-S} + \varepsilon^2 q_\alpha^{B} + \cdots, \end{aligned} \tag{10}$$

where upper indexes correspond to Navier-Stokes and Burnett terms. These terms are discussed in detail in [2], some formulas from [2] will be used below.

Coming back to the abstract equation (1) we can now specify the operators A, B and C in (1) and discuss a possible choice of regularizing operator R. It is quite clear that we should restrict a class of such operators by using some reasonable considerations. In particular, the following two conditions seem quite natural:

(A) Modified equations should have a general form of equations of hydrodynamics (7) with some new fluxes $\Pi_{\alpha\beta}$ and Q_α instead of $\pi_{\alpha\beta}$ and q_α respectively.
(B) Modified equations should preserve all "principal properties" of original equations.

The condition (B) will be discussed in the next section, here we concentrate on the condition (A).

There is also another argument , which is not easy to formalize. It is obviously preferable to choose "the simplest" class of transformations R (change of variables) provided this class is sufficient to achieve our goals. Therefore we shall not consider *all* possible equivalent transformations of (7). Instead we specify a particular class which seems to be the simplest one.

Proposition 1 *The transformation of* (7)

$$\rho' = \rho, \qquad u' = u, \qquad T' = T + \frac{1}{\rho}\operatorname{div} R \tag{11}$$

with arbitrary (smooth) vector function $R(x,t)$, leads to the following equations for ρ', u' and T' (primes are omitted below):

$$\begin{aligned} &\rho_t + \operatorname{div} \rho u = 0, \qquad \rho \mathcal{D} u_\alpha + \frac{\partial p}{\partial x_\alpha} + \frac{\partial \Pi_{\alpha\beta}}{\partial x_\beta} = 0, \\ &\frac{3}{2}\rho \mathcal{D} T + p \operatorname{div} u + \Pi_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} + \operatorname{div} Q = 0, \end{aligned} \tag{12}$$

where

$$\Pi = \widetilde{\pi} - I \operatorname{div} R, \qquad Q = \widetilde{q} - \frac{3}{2}[\mathcal{D} R + R \operatorname{div} u - (R \cdot \nabla) u], \tag{13}$$

I denotes the identity tensor, tildes mean that π and q are expressed in new variables.

Proof is given by a straightforward calculation.

The above transformation is a particular (simplified) version of more general transformations from [2]. Note that $\Pi_{\alpha\beta} = \Pi_{\beta\alpha}$, but

$$\Pi_{\alpha\alpha} = Tr\, \Pi = -3 \operatorname{div} R \neq 0 \tag{14}$$

in (12). Thus, the transformation (11) preserves the form of the original equations (7), however it does not preserve the identity $Tr\, \pi = 0$.

Our goal now is to specify the vector function $R(x,t)$ in order to obtain a reasonable generalization of Burnett equations. This will be done in Sect. 3.

3 Generalized Burnett Equations

In order to satisfy the above condition (B) we need to specify Navier-Stokes and Burnett terms in (10). It is well-known that

$$\pi_{\alpha\beta}^{N-S} = -2\mu(T)\overline{\frac{\partial u_\alpha}{\partial x_\beta}}, \qquad q_\alpha^{N-S} = -\lambda(T)\frac{\partial T}{\partial x_\alpha}, \tag{15}$$

where $\mu(T)$ and $\lambda(T)$ denote respectively the coefficients of viscosity and heat conductivity, and

$$\overline{a_{\alpha\beta}} = \frac{1}{2}\left(a_{\alpha\beta} + a_{\beta\alpha} - \frac{2}{3}\delta_{\alpha\beta} Tr\, a\right). \tag{16}$$

The Burnett terms are less known. We use the notations of [2] and present them in the following form:

$$\pi_{\alpha\beta}^{B} = -\frac{1}{\rho}\left\{\frac{T A(T)}{\rho}\overline{\frac{\partial^2 \rho}{\partial x_\alpha \partial x_\beta}} + [A(T) - C(T)]\overline{\frac{\partial^2 T}{\partial x_\alpha \partial x_\beta}} + G_{\alpha\beta}\right\}, \tag{17}$$

$$q_\alpha^{B} = -\frac{T}{2\rho}\left\{\frac{1}{3}[4B(T) - C(T)]\frac{\partial}{\partial x_\alpha}\operatorname{div} u - C(T)\Delta u_\alpha + G_\alpha\right\}, \tag{18}$$

where $A(T)$, $B(T)$ and $C(T)$ are Burnett coefficients introduced in [2], $G_{\alpha\beta}$ and G_α are bilinear forms on derivatives

$$\nabla \log \rho, \qquad \nabla \log T; \qquad \frac{\partial u_i}{\partial x_j}, \quad i, j = 1, 2, 3; \tag{19}$$

with coefficients that depend only on T. The coefficients A, B and C are completely defined by the Chapman-Enskog function (see (9)) in the notation of [2]

$$F_0 = M\left[\frac{1}{T}\frac{\partial u_\alpha}{\partial x_\beta}\varphi_{\alpha\beta}(c) + \frac{1}{T^2}\frac{\partial T}{\partial x_\alpha}\varphi_\alpha(c)\right], \quad c = v - u. \tag{20}$$

It was shown in [2] that

$$A(T) = \frac{1}{5T}\left\langle M\varphi_{\alpha\beta}, \varphi_{\alpha\beta}\right\rangle, \qquad B(T) = \frac{1}{3T^2}\left\langle M\varphi_\alpha, \varphi_\alpha\right\rangle, \qquad C(T) = \frac{1}{5T^2}\left\langle Mc_\alpha\varphi_\beta, \varphi_{\alpha\beta}\right\rangle. \tag{21}$$

Note also that

$$\mu(T) = -\frac{1}{10T}\left\langle M\varphi_{\alpha\beta}, c_\alpha c_\beta\right\rangle, \qquad \lambda(T) = -\frac{1}{6T^2}\left\langle M\varphi_\alpha, |c|^2 c_\alpha\right\rangle. \tag{22}$$

The following formulas (exact for Maxwell molecules) are convenient for various estimates:

$$A \approx 2\frac{\mu^2}{T}, \qquad B \approx \frac{45\mu^2}{8T}, \qquad C \approx 3\frac{\mu^2}{T}, \qquad \lambda \approx \frac{15}{4}\mu. \tag{23}$$

The terms $G_{\alpha\beta}$ and G_α in (17), (18) will be clarified below. They depend bilinearly on first derivatives (19).

Coming back to the general scheme (see (1)–(3)) of the regularization of Burnett equations by transformation (2), we can now specify the vector $R(x,t)$ in (11). In accordance with the condition (B), the symmetric tensor Π and the vector Q in (13) must have the same structure as corresponding Burnett terms. This leads to a unique choice of $R(x,t)$ in the following form:

$$R(x,t) = \varepsilon^2\frac{S(x,t)}{\rho}, \quad S = a(T)\bigtriangledown\log\rho + b(T)\bigtriangledown\log T, \tag{24}$$

with indefinite coefficients $a(T)$ and $b(T)$.

Indeed the vector function $R(x,t)$ must be: (1) a vector; (2) a linear form on derivatives (19); (3) a homogeneous function of ρ of order $m = -1$. The only vector linear on derivatives of u is $\operatorname{rot} u$, which is in fact a pseudo-vector. Therefore we reject it and obtain (24). Note that a and b cannot depend on u because of Galilei invariance.

Hence, the uncertainty in the regularizing transformation (11) is reduced to two unknown functions $a(T)$ and $b(T)$. As we shall see below, it is convenient to introduce two dimensionless parameters $\theta_{1,2}$ in the following way:

$$\begin{aligned} a(T) &= \frac{2}{3}[-TA(T) + \theta_1 g(T)], \\ b(T) &= \frac{2}{3}T[C(T) - A(T)] + \theta_2 g(T), \\ g &= \frac{T}{3}(5A + 2B - 4C). \end{aligned} \tag{25}$$

Generally speaking, this is just a simple replacement of $a(T)$, $b(T)$ by new unknown functions $\theta_{1,2}(T)$. Let us consider, however, a particular case of power-like intermolecular potentials (or hard spheres). Then $A(T)$, $B(T)$ and $C(T)$ are proportional to T^γ (for example, $\gamma = 1$ for hard spheres) and $g(T)$ is proportional to $T^{\gamma+1}$. Hence, the only way to preserve

properties of original Burnett equations, is to choose $\theta_{1,2}(T)$ in (25) independently of T. Thus, the parameters $\theta_{1,2}$ are simply numbers for all power-like potentials. Therefore the most natural choice of unknown $a(T)$ and $b(T)$ is given by (25), where $\theta_{1,2}$ are numbers. All these considerations can be completely formalized by demanding that the new equations (12) preserve all group invariant transformations of the original Burnett equations.

Hence, the uncertainty in the transformation (11), (24) is reduced to two numerical parameters $\theta_{1,2}$ in (25). The parameters should be chosen from conditions of hyperbolicity and stability (see Sect. 4). As we shall see below, it is convenient to introduce a third parameter θ_3 in such a way that

$$\theta_1 + \theta_2 + \theta_3 = 1. \tag{26}$$

Then we can formulate a definition of the whole class of generalized Burnett equations (GBEs). In order to avoid complicated notations with primes (see (11)) we do this in the following way.

Definition 1 Generalized Burnett equations, denoted by a symbol GBEs $(\theta_1, \theta_2, \theta_3)$ are equations for *auxiliary* variables (ρ, u, T) such that the *true* hydrodynamical variables are expressed through (ρ, u, T) by equalities

$$\rho^{\mathrm{tr}} = \rho, \qquad u^{\mathrm{tr}} = u, \qquad T^{\mathrm{tr}} = T - \frac{\varepsilon^2}{\rho} \operatorname{div} \frac{S}{\rho}, \tag{27}$$

(see (11), (24)).

Any two parameters of $\theta_{1,2,3}$ can be chosen arbitrarily, then the third parameter can be found from (26); the functions $a(T)$ and $b(T)$ in (24) are given (for fixed $\theta_{1,2}$) in (25). The GBEs $(\theta_1, \theta_2, \theta_3)$ are, by definition, (12), where

$$\Pi_{\alpha\beta} = \varepsilon \pi^{N-S}_{\alpha\beta} + \varepsilon^2 \left(\pi^{B}_{\alpha\beta} - \delta_{\alpha\beta} \operatorname{div} \frac{S}{\rho} \right), \tag{28}$$

$$Q_\alpha = \varepsilon q^{N-S}_\alpha + \varepsilon^2 q^B_\alpha + \frac{\varepsilon^2}{\rho} \left\{ 3 S_\beta \overline{\frac{\partial u_\alpha}{\partial x_\beta}} + \left(\frac{3}{2} a + b \right) \frac{\partial}{\partial x_\alpha} \operatorname{div} u \right.$$
$$\left. + (\operatorname{div} u) \left[(a' T - 2a) \frac{\partial}{\partial x_\alpha} \log \rho + (b' T - 2b) \frac{\partial}{\partial x_\alpha} \log T \right] \right\}, \quad \alpha, \beta = 1, 2, 3, \tag{29}$$

where primes denote differentiation with respect to T.

Remark 1 The above formula for Q_α follows by straightforward calculations from the corresponding formula in (13) provided the term $\mathcal{D}(S/\rho)$ is computed in the Euler approximation, i.e. for $\varepsilon = 0$.

4 Hyperbolicity and Stability of GBEs

Our goal in this section is to determine a domain of parameters (θ_1, θ_2) in $\mathbb{R}^2$ such that the corresponding GBEs $(\theta_1, \theta_2, \theta_3)$ are hyperbolic and stable. By stability we mean the following property: any constant stationary solution $\{\rho = \rho_0 > 0, T = T_0 > 0, u = 0\}$ is stable with respect to small perturbations (it is sufficient to consider $u = 0$, since the equations are Galilei-invariant). In fact, the stability implies hyperbolicity, as we shall see below.

In order to study the hyperbolicity it is sufficient to consider just terms with higher derivatives in GBEs $(\theta_1, \theta_2, \theta_3)$. By using (11), (28), (29), (17), (18), (24), (25), we obtain

$$\begin{aligned} &\rho_t = \cdots, \qquad \rho u_t = \varepsilon^2 \frac{g(T)}{\rho}\left[\frac{2\theta_1}{3\rho} \triangle (\nabla\rho) + \frac{\theta_2}{T} \triangle (\nabla T)\right] + \cdots, \\ &\frac{3}{2}\rho T_t = \varepsilon^2 \frac{g(T)}{\rho}\theta_3 \triangle \operatorname{div} u + \cdots; \qquad \theta_1 + \theta_2 + \theta_3 = 1, \end{aligned} \tag{30}$$

where dots denote terms with lower derivatives. The reader can now see that our way of parametrization leads to a simple form of terms with third derivatives. The hyperbolicity condition reads

$$\theta_2\theta_3 = \theta_2(1 - \theta_1 - \theta_2) \geq 0. \tag{31}$$

Note that the classical Burnett equations are also a particular case of GBE $(\theta_1, \theta_2, \theta_3)$ with parameters (generally speaking, functions of T)

$$\theta_1^B = \frac{T A(T)}{g(T)}, \qquad \theta_2^B = \frac{2}{3} T \frac{A(T) - C(T)}{g(T)}, \tag{32}$$

which corresponds to $a = b = 0$ in (25).

In the approximation (23) we obtain

$$g(T) \approx \frac{37}{12}\mu^2, \qquad \theta_1^B \approx \frac{24}{37}, \qquad \theta_2^B \approx -\frac{24}{111}, \tag{33}$$

i.e. these equations are not hyperbolic, as we already knew.

We assume that the condition (31) is satisfied and consider the question of stability. The Knudsen number ε is just a formal parameter in our equations, we can assume that $\varepsilon = 1$ without loss of generality. Then we should consider GBEs $(\theta_1, \theta_2, \theta_3)$ in the neighborhood of the constant solution $\{\rho = \rho_0 > 0,\ T = T_0 > 0,\ u = 0\}$. It is important that all other Burnett terms (i.e. terms of order $O(\varepsilon^2)$), which are omitted in (30), are quadratic with respect to derivatives. Therefore such terms do not contribute to linearized equations. Hence, the linearized GBEs contain all usual terms of linearized Navier-Stokes equations plus Burnett terms with third derivatives (see (30)).

We denote

$$\begin{aligned} &T = T_0(1 + \widetilde{T}), \qquad \rho = \rho_0(1 + \widetilde{\rho}), \qquad u = T_0^{1/2}\widetilde{u}, \\ &x = l\widetilde{x}, \qquad t = \tau\widetilde{t}; \qquad \tau = \frac{\mu(T_0)}{\rho_0 T_0}, \qquad l = \tau T_0^{1/2}, \end{aligned} \tag{34}$$

and obtain the following linearized equations (all variables below are dimensionless, tildes are omitted)

$$\begin{aligned} &\rho_t + \operatorname{div} u = 0, \qquad u_t + \nabla\rho + \nabla T = \triangle u + \frac{1}{3}\nabla \operatorname{div} u + g_0\left[\frac{2\theta_1}{3} \triangle (\nabla\rho) + \theta_2 \triangle (\nabla T)\right], \\ &T_t + \frac{2}{3}\operatorname{div} u = \frac{5}{2}\gamma_0 \triangle T + \frac{2}{3} g_0\theta_3 \triangle \operatorname{div} u, \qquad \theta_1 + \theta_2 + \theta_3 = 1, \end{aligned} \tag{35}$$

where

$$\gamma_0 = \frac{4\lambda(T_0)}{15\mu(T_0)}, \qquad g_0 = \frac{T_0}{3\mu^2(T_0)}[5A(T_0) + 2B(T_0) - 4C(T_0)]. \tag{36}$$

Then we make the usual substitution

$$(\rho, u, T) \sim \exp[\lambda(\kappa)t + i\kappa \cdot x], \quad \kappa \in \mathbb{R}^3, \tag{37}$$

(the standard notation for eigenvalues $\lambda(\kappa)$ should not be confused with similar notation $\lambda(T)$ for the heat-conduction coefficient) and obtain a dispersion relation for $\lambda(\kappa)$ (a cubic equation). We omit elementary calculations and present the resulting equation:

$$\begin{aligned} &\lambda^3 + a_1(z)\lambda^2 + a_2(z)\lambda + a_3(z) = 0, \quad z = |\kappa|^2, \\ &a_1 = \left(\frac{4}{3} + \frac{5}{2}\gamma_0\right)z, \quad a_2 = \frac{z}{3}[5 + 2(5 + g_0)z + 2g_0^2\theta_2\theta_3 z^2], \\ &a_3 = \frac{5}{2}\gamma_0 z^2\left(1 + \frac{2}{3}\theta_1 g_0 z\right). \end{aligned} \tag{38}$$

The stability condition reads

$$\operatorname{Re}\lambda < 0 \quad \text{for all } z > 0. \tag{39}$$

This is equivalent to the following conditions on coefficients [5]:

$$\{a_i > 0, i = 1, 2, 3; a_1 a_2 > a_3\} \quad \text{for all } z > 0. \tag{40}$$

The inequality $a_2 > 0$ obviously implies (for large z) the hyperbolicity condition (31). The domain of stability in the plane of parameters (θ_1, θ_2) depends on γ_0 and g_0 (note that $\gamma_0 > 0$ for any molecular model). A realistic estimate (exact for Maxwell molecules)

$$\gamma_0 \approx 1, \qquad g_0 \approx \frac{37}{12} \tag{41}$$

follows from (23). Therefore it seems reasonable to assume that $g_0 > 0$ for realistic models. Then we obtain necessary conditions of stability

$$\theta_1 \geq 0, \quad \theta_2(1 - \theta_1 - \theta_2) \geq 0, \tag{42}$$

which guarantee that $a_{1,2,3} > 0$. The remaining condition $a_1 a_2 > a_3$ leads to the quadratic inequality

$$\begin{aligned} &P(z) = z^2 + 2\alpha z + \beta > 0, \quad z > 0, \\ &\alpha = \frac{1}{2\Delta}[(5 + g_0)(8 + 15\gamma_0) - 15\gamma_0 g_0\theta_1], \quad \beta = \frac{5(4 + 3\gamma_0)}{\Delta}, \\ &\Delta = (8 + 15\gamma_0)g_0^2\theta_2(1 - \theta_1 - \theta_2), \end{aligned} \tag{43}$$

provided $\Delta > 0$. If $\Delta = 0$ (i.e. $\theta_2 = 0$ or $\theta_2 = 1 - \theta_1$) then necessary and sufficient conditions are:

$$\theta_1 \geq 0, \quad (5 + g_0)(8 + 15\gamma_0) - 15\gamma_0 g_0\theta_1 \geq 0. \tag{44}$$

The same inequalities for θ_1 yield the sufficient conditions if $\Delta > 0$. In this case, however, there is one more region of stability in the plane (θ_1, θ_2) for such θ_1, that

$$2\alpha\Delta = (5 + g_0)(8 + 15\gamma_0) - 15\gamma_0 g_0\theta_1 < 0. \tag{45}$$

Then a simple analysis of the inequality (43) shows that it reduces to $\alpha^2 < \beta$. Thus, we obtain

$$\theta_2^2 + \theta_2(\theta_1 - 1) + \frac{[(5 + g_0)(8 + 15\gamma_0) - 15\gamma_0 g_0 \theta_1]^2}{20(4 + 3\gamma_0)(8 + 15\gamma_0)g_0^2} < 0 \tag{46}$$

for θ_1 satisfying the inequality (45). This is a complete description of the stable domain in the plane (θ_1, θ_2) for arbitrary g_0 and γ_0. It is easy to show that such a domain always has an infinite measure (area). Having in mind estimates (41), we can assume that $g_0 \geq -5$ for any realistic potential. Then the simple triangle region

$$\theta_{1,2} \geq 0, \quad \theta_1 + \theta_2 \leq 1 \tag{47}$$

is always stable and sufficient for practical applications.

Finally we describe the whole domain of stability for Maxwell molecules, when (23) are exact. Then we obtain:

$$\begin{aligned} &\text{(a)} \quad 0 \leq \theta_1 \leq 1, \quad 0 \leq \theta_2 \leq 1 - \theta_1; \\ &\text{(b)} \quad 1 < \theta_1 \leq \theta_*, \quad 1 - \theta_2 \leq \theta_2 < 0; \\ &\text{(c)} \quad \theta_1 > \theta_*, \quad \theta_2^- \leq \theta_2 \leq \theta_2^+; \end{aligned} \tag{48}$$

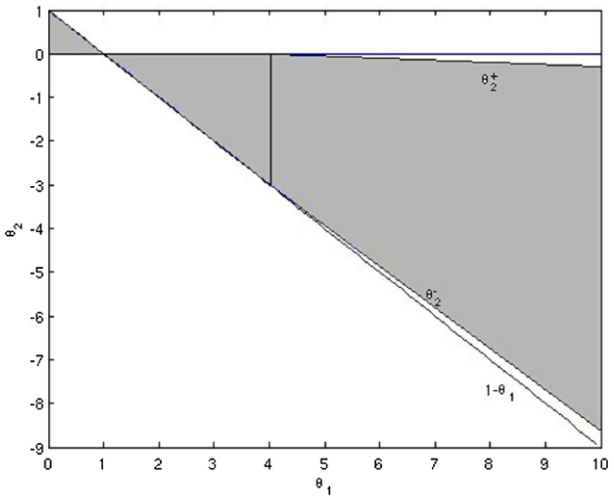
$$\theta_* = \frac{2231}{555} \approx 4.02, \quad \theta_2^{\pm} = \frac{1}{2}\left[1 - \theta_1 \pm \sqrt{(\theta_1 - 1)^2 - \frac{45}{161}(\theta_1 - \theta_*)^2}\right].$$

We also note that the domain

$$\theta_1 < 0, \quad 0 \leq \theta_2 \leq 1 - \theta_1 \tag{49}$$

(not shown in Fig. 1) is hyperbolic, but unstable. The domain of hyperbolicity (31) for $0 \leq \theta_1 \leq \theta_*$ coincides with the domain of stability. A difference between these two domains remains very small for $\theta_* < \theta_1 \leq 10$ (Fig. 1). Classical Burnett equations correspond to the point (33).

Fig. 1



It was already mentioned, that the triangle (47) seems to be sufficient for practical applications with any typical molecular model (for example, hard spheres). We just need to verify the condition of stability

$$g_0 = \frac{T}{3\mu^2(T)}[5A(T) + 2B(T) - 4C(T)] \geq -5 \tag{50}$$

for the triangle (47) (note that g_0 is a number for hard spheres and power-like potentials).

5 Explicit Form of GBEs

The aim of this section is to present GBEs $(\theta_1, \theta_2, \theta_3)$ in the form convenient for applications. In order to do this we pass to the dimensionless quantities in (21), (24), (25) by denoting

$$\begin{aligned} \{A(T), B(T), C(T)\} &= \frac{\mu^2(T)}{T}\{\widetilde{A}, \widetilde{B}, \widetilde{C}\}, \\ \{a(T), b(T), g(T)\} &= \mu^2(T)\{\widetilde{a}, \widetilde{b}, \widetilde{g}\}, \end{aligned} \tag{51}$$

where $\mu(T)$ is the viscosity coefficient (22). In the sequel of the paper these notations are used *without tildes*. Then the dimensionless coefficients $\{A, B, C, a, b, g\}$ where

$$g = \frac{1}{3}(5A + 2B - 4C), \tag{52}$$

are numbers for any power-like (and hard sphere) intermolecular potential, moreover

$$\mu(T) = \text{const}.T^{\omega} \tag{53}$$

in such cases ($\omega = 1$ for Maxwell molecules, $\omega = 1/2$ for hard spheres). Otherwise we assume that $\{A, B, C, a, b, g\}$ are slowly varying functions of T and use (53) below, with $\omega = \omega(T)$.

Then the two-parameter family of generalized Burnett equations $\{\text{GBEs}(\theta_1, \theta_2, \theta_3), \theta_1 + \theta_2 + \theta_3 = 1\}$ is described in the following way. We choose the classical Burnett equations (7), (10) (dots are replaced by zeroes in (10)) as a starting point. Then we fix a pair of real parameters (θ_1, θ_2) and denote

$$a = \frac{2}{3}(-A + \theta_1 g), \qquad b = \frac{2}{3}(C - A) + \theta_2 g, \tag{54}$$

where the dimensionless coefficients A, C, g (see (53)) are known from the initial Burnett equations. For example,

$$A = 2, \quad B = 3, \quad g = \frac{37}{12} \tag{55}$$

for Maxwell molecules, see (23).

As explained in Sect. 3, GBEs $(\theta_1, \theta_2, \theta_3)$ are equations for *auxiliary* variables (ρ, u, T), which are connected with *true* hydrodynamical variables by equalities

$$\begin{aligned} &\rho^{\text{tr}} = \rho, \qquad u^{\text{tr}} = u, \qquad \rho T^{\text{tr}} = P(\rho, T) = \rho T - \varepsilon^2 \,\text{div}\, \frac{\mu^2(T)}{\rho} S, \\ &S = a\nabla \log \rho + b\nabla \log T, \end{aligned} \tag{56}$$

in the notation of (54). The equation for (ρ, u, T) read

$$\begin{aligned} &\rho_t + \operatorname{div} \rho u = 0, \qquad \rho \mathcal{D} u_\alpha + \nabla_\alpha P(\rho, T) + \frac{\partial \Pi_{\alpha\beta}}{\partial x_\beta} = 0, \\ &\frac{3}{2}\rho \mathcal{D} T + P(\rho, T) \operatorname{div} u + \Pi_{\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta} + \operatorname{div} Q = 0, \end{aligned} \tag{57}$$

where

$$Q_\alpha = q_\alpha + \frac{\mu^2(T)}{\rho}\left[3S_\beta \overline{\frac{\partial u_\alpha}{\partial x_\beta}} + 2(\omega - 1)S_\alpha \operatorname{div} u + \Gamma_\alpha + \left(\frac{3}{2}a + b\right)\nabla_\alpha \operatorname{div} u\right], \tag{58}$$

q_α and $\Pi_{\alpha\beta}$ are the usual fluxes (10) from the Burnett equations; P, S and ω are given in (56), (55). If $\omega = \text{const.}$ (power-like potentials and hard spheres), then a and b are numbers and $\Gamma_\alpha = 0$ in (58). Otherwise we obtain

$$\Gamma_\alpha = T\left[\frac{a'(T)}{a(T)}\nabla_\alpha \log \rho + \frac{b'(T)}{b(T)}\nabla_\alpha \log T\right]. \tag{59}$$

The above formulas (together with conditions of stability from Sect. 4) yield the complete information for practical use of GBEs, though the important question of boundary conditions is not considered in this paper. Main results are summarized in Conclusions.

6 Conclusions

The two-parameter family of generalized Burnett equations {GBEs $(\theta_1, \theta_2, \theta_3)$, $\theta_1 + \theta_2 + \theta_3 = 1$}, seems to be the most natural replacement for classical (unstable) Burnett equations. The equations are stable (see the beginning of Sect. 4) and hyperbolic in the wide domain of parameters. In particular, they are stable for all $0 \le \theta_{1,2,3} \le 1$ for typical molecular models satisfying (50). Their non-uniqueness seems unavoidable unless some new restrictions are introduced.

The advantage of GBEs (as compared, for example, with hyperbolic Burnett equations from [2]) is that they have the usual form of hydrodynamic equations and are not more complicated than original Burnett equations. In fact, GBEs contain the same terms, however with other coefficients. Some of the equations, e.g. GBEs (0, 0, 1), GBEs (0, 1, 0) and GBEs (1, 0, 0) are even simpler: the number of third derivatives is reduced.

A freedom of choice of parameters (θ_1, θ_2) can also be considered as an advantage of these equations: they can be different for different specific problems. Optimal choice of parameters and corresponding boundary conditions needs further investigations.

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